The resultant wave (solid line) is obtained by adding the two waves graphically. We begin in Fig. 1-9(*a*) with zero phase angle between the two waves, i.e. the two waves are completely in phase with each other. The resultant amplitude is equal to twice the amplitude of the given waves.

Fig. 1-9(*b*) shows the addition of two identical waves with 18° phase difference between them. Similarly, Fig. 1-9(*c*) to Fig. 1-9(*j*) are the resultants of the additions of two identical waves with progressively greater values of phase angle between the two identical waves.

When the two identical waves are completely out of phase, i.e. the phase angle between the two waves is 180°, the resultant wave is zero. In other words, the two waves cancel each other.

SIMPLE HARMONIC MOTIONS

1.8. A simple harmonic motion is given as $x(t) = 10 \sin(10t - 30°)$ where x is measured in meters, t in seconds, and the phase angle in degrees. Find (*a*) the frequency and period of the motion, (*b*) the maximum displacement, velocity and acceleration, (*c*) the displacement, velocity and acceleration at $t = 0$ and $t = 1$ seconds.

(*a*)
$$x(t) = 10 \sin(10t - 30°) = A_0 \sin(\omega t - \theta)$$

Then $\omega = 10$ rad/sec, $f = \omega/2\pi = 1.6$ cyc/sec, and $p = 1/f = 0.63$ sec.

(*b*) Displacement is $x(t) = 10 \sin(10t - 30°)$. Thus the maximum displacement is 10 m.

Velocity is $dx/dt = \omega A_0 \cos(\omega t - \theta)$. Thus the maximum velocity is $10(10) = 100$ m/sec.

Acceleration is $d^2x/dt^2 = -\omega^2 A_0 \sin(\omega t - \theta)$, and so the maximum acceleration is $-10^2(10) = -1000$ m/sec^2.

(*c*) At $t = 0$:

$$x(0) = 10 \sin(-30°) = 10(-0.5) = -5 \text{ m}$$
$$\dot{x}(0) = \omega A_0 \cos(-30°) = 10(10)(0.866) = 86.6 \text{ m/sec}$$
$$\ddot{x}(0) = -\omega^2 A_0 \sin(-30°) = -(10)^2(10)(-0.5) = 500 \text{ m/sec}^2$$

At $t = 1$:

$$x(1) = 10 \sin(10 - 30°) \doteq 10 \sin(570° - 30°) \doteq 10 \sin 180° = 0$$
$$\dot{x}(1) = 10(10) \cos 180° = -100 \text{ m/sec}$$
$$\ddot{x}(1) = -(10)^2(10) \sin 180° \doteq 0$$

FREE VIBRATION

1.9. Determine the differential equation of motion and natural frequency of vibration of the simple single-degree-of-freedom spring-mass system shown in Fig. 1-10.

Apply Newton's law of motion, $\Sigma F = ma$. For vertical motion, the forces acting on the mass are the spring force $k(\delta_{st} + x)$ and the weight mg of the mass. Therefore the differential equation of motion is

$$m\ddot{x} = -k(\delta_{st} + x) + mg$$

where δ_{st} is the static deflection due to the weight of the mass acting on the spring. Then $mg = \delta_{st}k$, and the equation of motion becomes

$$m\ddot{x} + kx = 0$$

which is the differential equation for SHM. The general forms of solution for this equation are

$$x(t) = A \sin\sqrt{k/m}\, t + B \cos\sqrt{k/m}\, t$$
$$x(t) = C \sin(\sqrt{k/m}\, t + \phi)$$
$$x(t) = D \cos(\sqrt{k/m}\, t - \theta)$$

where A, B, C, D, ϕ and θ are arbitrary constants depending on initial conditions $x(0)$ and $\dot{x}(0)$. Two constants must appear in each of the general solutions because this is a second order differential equation.

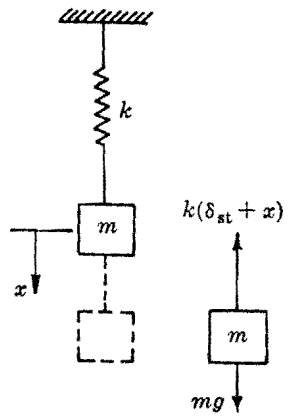

Fig. 1-10

For an initial displacement $x(0) = x_0$ and zero initial velocity $\dot{x}(0) = 0$, we have $A = 0$, $B = x_0$ and hence

$$x(t) = x_0 \cos \sqrt{k/m}\, t$$

Physically, this solution represents an undamped free vibration, one cycle of which occurs when $\sqrt{k/m}\, t$ varies through 360 degrees. Therefore the period P and the natural frequency f_n are

$$P = \frac{2\pi}{\sqrt{k/m}} \text{ sec} \quad \text{and} \quad f_n = 1/P = \frac{\sqrt{k/m}}{2\pi} \text{ cyc/sec}$$

where $\omega_n = \sqrt{k/m}$ rad/sec is the circular natural frequency of the system.

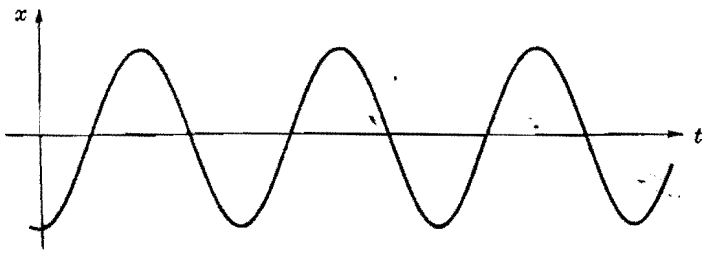

Fig. 1-11. Free vibration without damping

1.10. A generalized single-degree-of-freedom spring-mass system with damping is shown in Fig. 1-12. Investigate its general motion.

Employing Newton's law of motion $\Sigma F = ma$,

$$m\ddot{x} = -c\dot{x} - kx \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = 0$$

where k is the spring constant, m the mass, and c the damping coefficient.

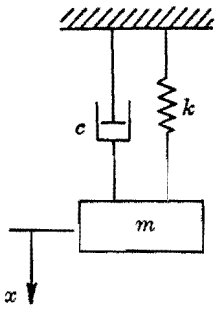

Fig. 1-12

We cannot assume solutions of the sine or cosine functions because of the term $c\dot{x}$. We assume $x = e^{rt}$; then $\dot{x} = re^{rt}$, $\ddot{x} = r^2e^{rt}$. Substituting these values into the differential equation of motion, we obtain

$$mr^2e^{rt} + cre^{rt} + ke^{rt} = 0 \quad \text{or} \quad mr^2 + cr + k = 0$$

The two values of r satisfying the above equation are

$$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$$

where $\omega_n = \sqrt{k/m}$, and $\zeta = c/2m\omega_n$ is called the damping factor. Thus the solution to the equation of motion is

$$x(t) = Ae^{r_1t} + Be^{r_2t}$$

where A and B are arbitrary constants determined by the two initial conditions imposed on the system.

Since the values of r depend on the magnitude of ζ, we have the following three cases of free vibration with damping:

Case 1: If ζ is greater than unity, the values of r are real and distinct; the amplitude of x is decreasing but will never change sign. Therefore oscillatory motion is not possible for the system regardless of initial conditions. This is *overdamped*, where

$$x(t) = Ae^{-r_1t} + Be^{-r_2t}$$

Case 2: If ζ is equal to unity, the values of r are real and negative, and are equal to $-\omega_n$. The motion of the system is again not oscillatory, and its amplitude will eventually reduce to zero. This is *critically-damped*, where

$$x(t) = (C + Dt)e^{-\omega_n t}$$

Case 3: If ζ is less than unity, the values of r are complex conjugates:

$$r_1 = \omega_n(-\zeta + i\sqrt{1-\zeta^2}), \qquad r_2 = \omega_n(-\zeta - i\sqrt{1-\zeta^2})$$

And if we define $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ as the damped natural frequency in rad/sec, we have

$$r_1 = -\zeta\omega_n + i\omega_d, \qquad r_2 = -\zeta\omega_n - i\omega_d$$

and

$$x(t) = e^{-\zeta\omega_n t}(Ee^{i\omega_d t} + Fe^{-i\omega_d t})$$

Expanding,

$$x(t) = e^{-\zeta\omega_n t}[(E+F)\cos\omega_d t + i(E-F)\sin\omega_d t]$$

Letting $E+F=G$ and $i(E-F)=H$, we finally obtain

$$x(t) = e^{-\zeta\omega_n t}(G\cos\omega_d t + H\sin\omega_d t)$$

As shown before, we may combine a cosine and sine function of the same frequency into a single sine or cosine function as

$$x(t) = Ie^{-\zeta\omega_n t}\sin(\omega_d t + \theta)$$

$$x(t) = Ie^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

where $I = \sqrt{G^2+H^2}$, $\theta = \tan^{-1}(G/H)$, $\phi = \tan^{-1}(H/G)$.

The motion is oscillatory with angular frequency ω_d. The amplitude of motion will decrease exponentially with time because of the term $e^{-\zeta\omega_n t}$, which is known as the *decaying factor*. This is underdamped vibration. Refer to Fig. 1-13.

Hence it may be concluded that the motion of a dynamic system with damping and having free vibration depends on the amount of damping present in the system. The resulting motion will be periodic only if the amount of damping present is less than critical, and the system oscillates with angular frequency slightly less than the free natural frequency of the system.

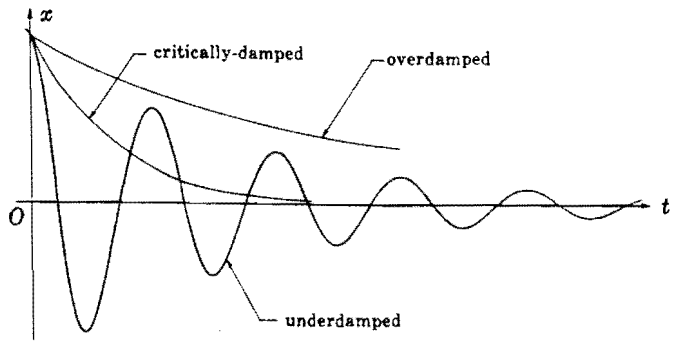

Fig. 1-13. Free vibration with damping

FORCED VIBRATION

1.11. Investigate the general motion of a simple spring-mass system with damping excited by a sinusoidal force $F_0\cos\omega t$ as shown in Fig. 1-14.

Employing Newton's law of motion,

$$m\ddot{x} = \text{sum of forces in the } x \text{ direction}$$
$$= -k(x+\delta_{st}) + mg - c\dot{x} + F_0\cos\omega t$$

But $k\delta_{st} = mg$, the weight of the mass; hence the equation of motion takes its most general form

$$m\ddot{x} + c\dot{x} + kx = F_0\cos\omega t$$

The general solution for this second order differential equation with constant coefficients is

$$x = x_c + x_p$$

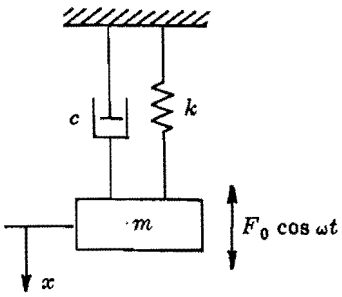

Fig. 1-14

where x_c is called the complementary solution, or the solution of the homogeneous equation, $m\ddot{x} + c\dot{x} + kx = 0$. x_p is the particular solution for the given equation.

The complementary solution, known as free vibration, has been solved previously in Problem 1-10. The particular solution, obtained from the nonhomogeneous part $F_0 \cos \omega t$ of the differential equation of motion, is

$$x_p(t) = A \sin \omega t + B \cos \omega t$$

and so

$$\dot{x}_p(t) = \omega A \cos \omega t - \omega B \sin \omega t$$

$$\ddot{x}_p(t) = -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t$$

Substituting these expressions into the equation of motion, we obtain

$$(kA - mA\omega^2 - c\omega B) \sin \omega t + (kB - mB\omega^2 + c\omega A) \cos \omega t = F_0 \cos \omega t$$

Equating the coefficients,

$$(k - m\omega^2)A - c\omega B = 0, \qquad c\omega A + (k - m\omega^2)B = F_0$$

from which

$$A = \frac{F_0 \omega c}{(k - m\omega^2)^2 + (c\omega)^2}, \qquad B = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2}$$

Then

$$x_p(t) = \frac{F_0 \omega c}{(k - m\omega^2)^2 + (c\omega)^2} \sin \omega t + \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2} \cos \omega t$$

We may combine these two sinusoidal functions of the same frequency either by rotating vectors or by trigonometric identities to obtain

$$x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \phi)$$

or

$$x_p(t) = \frac{F_0/k}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \cos(\omega t - \phi)$$

where $r = \omega/\omega_n$, $\omega_n = \sqrt{k/m}$, and $\phi = \tan^{-1} \dfrac{c\omega}{k - m\omega^2} = \tan^{-1} \dfrac{2\zeta r}{1 - r^2}$.

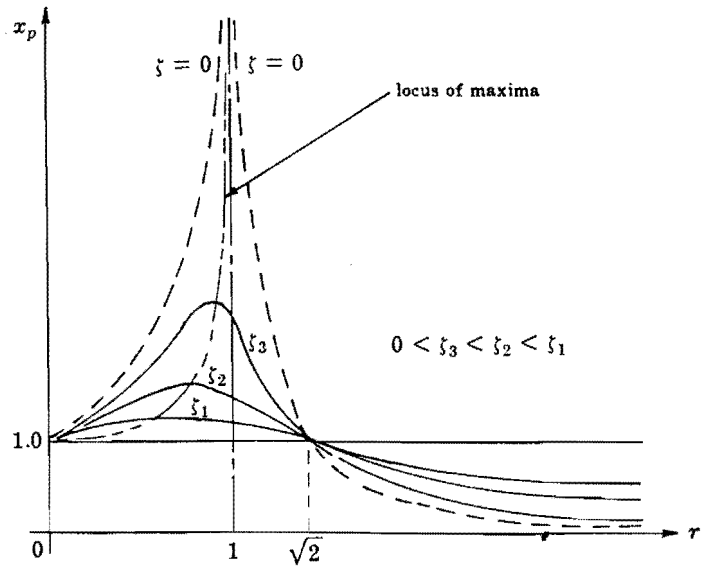

Fig. 1-15

Hence it may be concluded that the particular solution $x_p(t)$, which is known as the *steady state response* or *forced vibration*, is of the same frequency as that of the excitation force regardless of initial conditions. The amplitude of forced vibration depends on the amplitude and

frequency of the excitation force, and the parameters of the systems. At *resonance*, i.e. when the forcing frequency is equal to the natural frequency, or $\omega/\omega_n = 1$, the amplitude of forced vibration is limited only by the damping factor ζ and hence the amount of damping present. Therefore resonance should be avoided at all times. Finally, the steady state response of the system is not in phase with the excitation force; its variation by the phase angle ϕ is due to the presence of damping in the system. Without damping, the steady state response is either in phase or 180° out of phase with the excitation force. See Fig. 1-15 to Fig. 1-19.

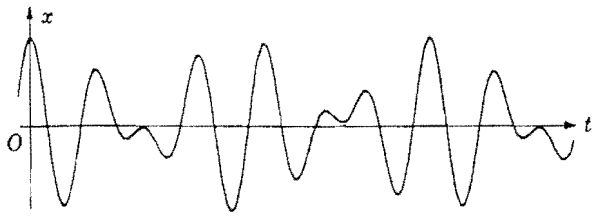

Fig. 1-16. Forced vibration without damping ($2f_n = 3f$)

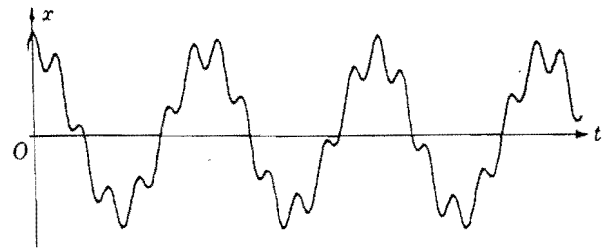

Fig. 1-17. Forced vibration without damping ($f_n = 6.28f$)

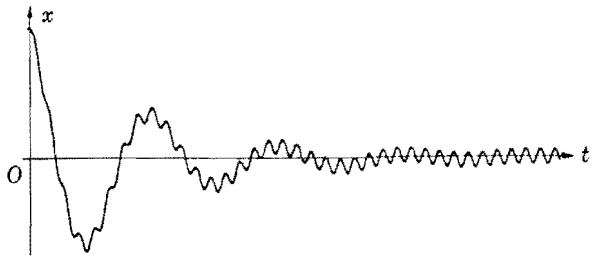

Fig. 1-18. Forced vibration with damping ($f_n = 6.28f$)

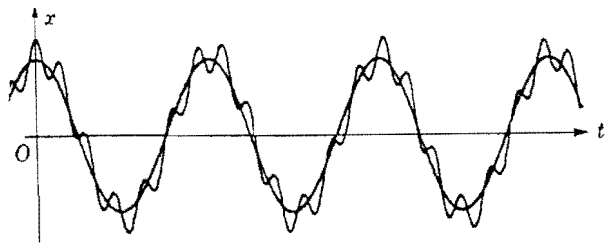

Fig. 1-19. Free and forced vibration ($f_n = 6.28f$)

ENERGY OF VIBRATION

1.12. Determine the power requirements for vibration testing and analysis.

In vibration testing, we have forced vibration. The work done is the product of the excitation and displacement, while power required is the rate of doing work. Let $F = F_0 \cos \omega t$ and $x = A \cos(\omega t - \phi)$; then the work done is

$$W = \int F\,dx = \int F_0 \cos \omega t[-A \sin(\omega t - \phi)\, d(\omega t)]$$

and work done per cycle of motion is

$$W = -F_0 A \int_0^{2\pi} \cos \omega t \sin (\omega t - \phi)\, d(\omega t)$$

as the angle ωt goes through a cycle of 2π. Since $\sin (\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$, the work done per cycle of motion becomes

$$\begin{aligned} W &= F_0 A \sin \phi \int_0^{2\pi} \cos^2 \omega t\, d(\omega t) - F_0 A \cos \phi \int_0^{2\pi} \cos \omega t \sin \omega t\, d(\omega t) \\ &= F_0 A \sin \phi \left[\frac{\omega t}{2} + \frac{\sin 2\omega t}{4}\right]_0^{2\pi} - F_0 A \cos \phi \left[\frac{\sin^2 \omega t}{2}\right]_0^{2\pi} \\ &= F_0 A \sin \phi \left[\frac{\omega t}{2} + \frac{\sin 2\omega t}{4}\right]_0^{2\pi} - F_0 A \cos \phi \left[\frac{1}{4} - \frac{\cos 2\omega t}{2}\right]_0^{2\pi} \\ &= \pi A F_0 \sin \phi \end{aligned}$$

If $F = F_0 \sin \omega t$ and $x = A \sin (\omega t - \phi)$, then the work done is

$$W = \int F\, dx = \int F \frac{dx}{dt}\, dt = \int F \dot{x}\, dt$$

The expression for work done in one cycle of motion is then

$$W = \int_0^{2\pi/\omega} F_0 \sin \omega t [\omega A \cos (\omega t - \phi)\, dt] = \int_0^{2\pi/\omega} F_0 A \omega \sin \omega t \cos (\omega t - \phi)\, dt$$

Since $\cos (\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$,

$$W = \int_0^{2\pi/\omega} \omega A F_0 \sin \omega t \cos \omega t \cos \phi\, dt + \int_0^{2\pi/\omega} \omega A F_0 \sin \phi \sin^2 \omega t\, dt$$

As shown earlier, the above expression can be reduced to

$$\begin{aligned} W &= \left[A F_0 \omega \cos \phi \frac{\sin^2 \omega t}{2} + A F_0 \omega \sin \phi \left(\frac{t}{2} - \frac{\sin 2\omega t}{4\omega}\right)\right]_0^{2\pi/\omega} \\ &= \left[A F_0 \omega \cos \phi \left(\frac{1}{4\omega} - \frac{\cos 2\omega t}{4\omega}\right) + A F_0 \omega \sin \phi \left(\frac{t}{2} - \frac{\sin 2\omega t}{4\omega}\right)\right]_0^{2\pi/\omega} \\ &= \pi A F_0 \sin \phi \end{aligned}$$

Thus the power required is proportional to the amplitude F_0 of the excitation force as well as to the amplitude A of the displacement. When there is no damping in the system, the work done by the driving force is zero because $\phi = 0°$ or $180°$. At resonance, energy is needed to build up the amplitude of vibration; and for this case, $\phi = 90°$.

1.13. The steady state response of a simple dynamic system to a sinusoidal excitation $10 \sin 0.1\pi t$ newtons is $0.1 \sin (0.1\pi t - 30°)$ meters. Determine the work done by the excitation force in (*a*) one minute and (*b*) one second.

(*a*) From Problem 1.12, the work done per cycle by the excitation force is given by

$$W = \int_0^{2\pi} F\, dx = \int_0^{P} F \dot{x}\, dt = \pi A F_0 \sin \phi$$

where $F_0 = 10$ newtons is the amplitude of the excitation force, $A = 0.1$ m is the amplitude of the steady state response, and $\phi = 30°$ is the phase angle. Hence work done by the excitation force is $W = 3.14(0.1)(10)(0.5) = 1.57$ joules/cyc. The angular frequency is 0.1π rad/sec and the period $P = 1/f = 20$ sec. In one minute, the excitation force will complete three cycles. Therefore work done by the excitation force in one minute is 4.71 joules.

(*b*) Work done per cycle is $W = \int_0^{20} F \dot{x}\, dt$. Then work done in one second is

$$W = \int_0^1 (10 \sin 0.1\pi t)(0.01\pi) \cos (0.1\pi t - 30°)\, dt = 0.05 \text{ joule}$$

1.14. Prove that the mean kinetic and potential energies of nondissipative vibrating systems are equal.

For free vibration without damping, the motion can be assumed harmonic and is given by

$$x(t) = A \sin \omega_n t$$

Kinetic energy $\mathrm{KE} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(\omega_n^2 A^2 \cos^2 \omega_n t) = \frac{1}{2}kA^2 \cos^2 \omega_n t$, where $\omega_n^2 = k/m$.

Potential energy $\mathrm{PE} = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \sin^2 \omega_n t$.

$$(\mathrm{KE})_{\mathrm{mean}} = \frac{1}{P}\int_0^P (\tfrac{1}{2}kA^2 \cos^2 \omega_n t)\, dt = \tfrac{1}{4}kA^2$$

$$(\mathrm{PE})_{\mathrm{mean}} = \frac{1}{P}\int_0^P (\tfrac{1}{2}kA^2 \sin^2 \omega_n t)\, dt = \tfrac{1}{4}kA^2$$

1.15. A uniform string fixed at both ends is displaced a distance h at the center and released from rest as shown in Fig. 1-20. Find the energy of transverse vibration of the string.

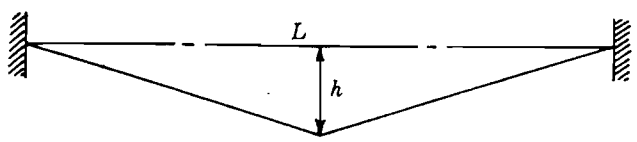

Fig. 1-20

The free transverse vibration of a uniform string can be expressed as

$$y(x, t) = \sum_{i=1,2,\ldots}^{\infty} A_i \sin \frac{i\pi x}{L} \cos\left(\frac{i\pi a}{L} t + \theta_i\right)$$

where A_i is the amplitude of motion and θ_i is the phase angle. (See Problem 1.17.) Then

$$\mathrm{KE} = \tfrac{1}{2}\rho_L \int_0^L \dot{y}^2\, dx = \frac{\rho_L a^2 \pi^2}{4L} \sum_{i=1,2,\ldots}^{\infty} i^2 A^2 \sin^2\left(\frac{i\pi a}{L} t + \theta_i\right)$$

$$\mathrm{PE} = \tfrac{1}{2}S \int_0^L \left[\frac{\partial^2 y}{\partial x^2}\right]^2 dx = \frac{S\pi^2}{4L} \sum_{i=1,2,\ldots}^{\infty} i^2 A_i^2 \cos^2\left(\frac{i\pi a}{L} t + \theta_i\right)$$

or

$$\mathrm{KE} + \mathrm{PE} = \frac{\pi^2 a^2 \rho_L}{4L} \sum_{i=1,2,\ldots}^{\infty} i^2 A_i^2$$

where S is the tension in the string, ρ_L is the mass per unit length of the string, and $a = \sqrt{S/\rho_L}$ is the speed of wave propagation.

From the initial conditions $\dot{y}(x, 0) = 0$ and $y(x, 0) = \begin{cases} 2hx/L, & 0 \le x \le L/2 \\ 2h(1 - x/L), & L/2 \le x \le L \end{cases}$ we obtain $A_i^2 = 64h^2/i^2\pi^2$. The expression for the energy of transverse vibration of the string becomes

$$\mathrm{KE} + \mathrm{PE} = 16\rho_L a^2 h^2/\pi^2 i^2 L, \qquad i = 1, 3, \ldots$$

Let the total energy associated with the fundamental mode of vibration be E_1, i.e.

$$E_1 = 16\rho_L a^2 h^2/L\pi^2$$

Then the energies associated with the first harmonic, second harmonic, third harmonic, ... are respectively

$$E_3 = E_1/9, \quad E_5 = E_1/25, \quad E_7 = E_1/49, \quad \ldots$$

Thus the main part of the energy of vibration is associated with the normal modes of low order. The quality of a tone is governed by the proportion of energy in each of the modes of vibration. Though the fundamental frequency may be the same, the energy distribution in the harmonics characterizes each musical instrument.