# Do we know the mass of a black hole? Mass of some cosmological black hole models

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Using a cosmological black hole model proposed recently, we have calculated the quasi-local mass of a collapsing structure within a cosmological setting due to different definitions put forward in the last decades to see how similar or different they are. It has been shown that the mass within the horizon follows the familiar Brown-York behavior. It increases, however, outside the horizon again after a short decrease, in contrast to the Schwarzschild case. Further away, near the void, outside the collapsed region, and where the density reaches the background minimum, all the mass definitions roughly coincide. They differ, however, substantially far from it. Generically, we are faced with three different Brown-York mass maxima: near the horizon, around the void between the overdensity region and the background, and another at cosmological distances corresponding to the cosmological horizon. While the latter two maxima are always present, the horizon mass maxima is absent before the onset of the central singularity.

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## I. INTRODUCTION

What does general relativity tell us about the mass of a cosmological structure in a dynamical setting? We know that massive sources produce gravitational field which has energy. In relativity theory, the equivalence of mass and energy means that it is only the combined energy which may be measured at a distance. We should, therefore, expect that because of non-linearity of the gravitational field the mass of the material source, its kinetic energy, and the gravitational energy it produces combine in a nonlinear and non-local way to produce the effective energy. In simplest case of spherical symmetry in vacuum, this effective energy is just the Schwarzschild energy. Although this argument about the effective energy seems very plausible, there are still disputes in the literature simply about the definition of mass in the spherically symmetric vacuum cases [4](for an interesting discussion about the concept of mass in relativity see also [3]). What if the material mass is embedded in a cosmological setting?

In a cosmological setting, due to the asymptotically non-flatness of the cosmological background, one can not use global definitions such as ADM mass and angular momentum [1]. This has led authors in the last decades to the notion of the quasi local mass (QLM) or quasi local energy(QLE), applicable to non-local structures in any general dynamical situations[2]. In this paper we will interchangeably use QLM or QLE as the same concept, although they may differ in other contexts. It is a fundamental fact of general relativity reflected in the equivalence principle that there is no such concept as the local mass of a local object: a local object at the origin of a freely falling frame will not experience any gravitational acceleration. In the absence of a local gravitational effect the notion of a QLM, corresponding to a two dimensional compact surface, although not uniquely defined, is the only concept one may try to apply to astrophysical cases.

Let us define a cosmological structure as an overdensity region within a cosmological background which we assume to be asymptotically FRW. There are not much viable exact models representing such a structure. We will rely on a recent analytical model proposed by the authors based on a inhomogeneous cosmological LTB model to construct such an asymptotically FRW universe including an overdensity region evolving to a black hole different from that of the Schwarzschild[6]. Our aim is to understand the notion of mass due to different definitions of QLM for such a cosmological structure within a FRW universe.

In general, there is no unique definition of the quasi local energy, except at infinity in an asymptotically flat space-time, where one has the Arnowitt-Deser-Misner [1] energy $E_{ADM}$ at the spatial infinity and the Bondi-Sachs [8] energy $E_{BS}$ at the null infinity. Hawking [12] defined a quasi local mass which has various desirable properties: it is zero for a metric sphere in flat spacetime, gives the correct mass for the Schwarzschild solution on a metric sphere, and tends to the Bondi mass asymptotically in a static asymptotically flat spacetime. However, it is non-zero for generic 2-surfaces in flat spacetime. Hayward [14] has proposed an expression for quasi local energy which maybe considered as a modification of Hawking's energy

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by shear and twist terms. Another attempt was due to Misner and Sharp [9] with a well-understood Newtonian limit [10].
The most promising QLM definition, however, seems to be the one proposed by Brown-York [15]. Motivated by the Hamiltonian formulation of general relativity, they found an interesting local quantity from which the definition of quasi local mass was extracted. Their definition depends, however, on the choice of the gauge along the 3-dimensional spacelike slice. It has the right asymptotic behavior but is not positive in general. Motivated by some geometric consideration, Liu-Yau [18] (see also Kijowski [19], Booth-Mann[16], and Epp [17]) introduced a mass which is gauge independent, and always positive. However, it was pointed out by O'Murchadha et al [20] that the Liu-Yau mass can be strictly positive even when the 2-surface is in a flat spacetime.
Our aim is to study some of these quasi local masses for spherically symmetric structures in a cosmological setting within general relativity to see how similar or different they are. We review different definitions of quasi local masses in section II, followed by the introduction of LTB metrics and the corresponding mass definitions in section III. In section IV we explicitly calculate numerically the masses for two cosmological toy black holes and a structure with an NFW mass profile [23]. We then conclude in section V discussing the results.

## II. QUASI LOCAL MASS DEFINITIONS

In general relativity the mathematical entity used to define the mass-energy is the symmetric energy momentum tensor $T_{\mu\nu}$, representing the source-current for gravity, although the proper interpretation of it is only the source density for gravity. This fact is one of the roots of difficulties to define mass in general relativity. Reasonable total energy-momentum can be associated with the whole space-time provided it is asymptotically flat. This has led general relativist to 'quasi-localization' of total quantities, and construction of 'quasi-local' mass-energy. Techniques used in the quasi localization depend on the actual form of the total quantities, yielding inequivalent definitions for the quasi-local masses [2]. Here we outline some of the mostly used definitions before going on to apply them to specific models for mass condensation within FRW cosmological models and try to interpret the results.

### A. Misner-Sharp mass

Take a collapsing ideal fluid within a compact spherically symmetric spacetime region described by the following metric in the comoving coordinates $(t, r, \theta, \varphi)$:

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R(t,r)^2 d\Omega^2. \quad (1)$$

assuming the energy momentum tensor for the perfect fluid in the form

$$T_t^t = -\rho(t,r), \quad T_r^r = p_r(t,r), \quad T_\theta^\theta = T_\varphi^\varphi = p_\theta(t,r), \quad (2)$$

with the week energy condition

$$\rho \geq 0, \quad \rho + p_r \geq 0, \quad \rho + p_\theta \geq 0, \quad (3)$$

we then obtain the Einstein equations in the form

$$\rho = \frac{2M'}{R^2 R'} \ , \quad p_r = -\frac{2\dot{M}}{R^2 \dot{R}}, \quad (4)$$

$$\nu' = \frac{2(p_\theta - p_r)}{\rho + p_r}\frac{R'}{R} - \frac{p_r'}{\rho + p_r}, \quad (5)$$

$$-2\dot{R}' + R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} = 0, \quad (6)$$

where

$$G = e^{-2\psi}(R')^2 \quad , \quad H = e^{-2\nu}(\dot{R})^2, \quad (7)$$

and $M$ is defined by

$$G - H = 1 - \frac{2M}{R}. \quad (8)$$

The function $M$ can also be written as

$$M = \frac{1}{2}\int_0^R \rho R^2 dR, \quad (9)$$

or

$$M = \frac{1}{8\pi}\int_0^r \rho\sqrt{(1 + (\frac{dR}{d\tau})^2 - \frac{2M}{R})}d^3V, \quad (10)$$

where

$$d^3V = 4\pi e^{\psi} R' dr, \quad (11)$$

and

$$\frac{d}{d\tau} = e^{\nu}\frac{d}{dt}. \quad (12)$$

The last form of the function $M$ indicates that when considered as energy, it includes contribution from the kinetic energy and the gravitational potential energy. $M$ is called the Misner-Sharp energy.
Hayward [10] showed that in the Newtonian limit of a perfect fluid, $M$ yields the Newtonian mass to the leading order and the Newtonian kinetic and potential energy to the next order. In vacuum, $M$ reduces to the Schwarzschild energy. At null and spatial infinity, $M$ reduces to the Bondi-Sachs and Arnowitt-Deser-Misner energies respectively [10].

### B. Hawking mass

Hawking [12] defined a quasi-local mass for the spacelike topological 2-sphere S:

$$E_H(S) = \sqrt{\frac{Area(S)}{16\pi G^2}}(1 + \frac{1}{2\pi}\oint_S \rho\rho' dS) =$$
$$\sqrt{\frac{Area(S)}{16\pi G^2}}(\oint(-\Psi_2 - \sigma\lambda + \Phi_{11} + \Lambda)dS), \quad (13)$$

where the spin coefficients $\rho$ and $\rho'$ measure the expansion of outgoing and ingoing light cones. For the definitions of $\Psi_2$, $\sigma$, $\lambda$, $\Phi_{11}$ and $\Lambda$ see [12]. The Hawking mass has various desirable properties: it is zero for a sphere in flat spacetime, gives the correct mass for the Schwarzschild solution on a metric sphere, and tends to the Bondi mass asymptotically in a static asymptotically flat spacetime. It is invariant under the boost gauge transformation. It is, however, non-zero for generic 2-surfaces in a flat spacetime (see [2] for more detail).

### C. Hayward mass

Hayward, using a 2+2 formulation of general relativity, gives the following definition for a quasi-local energy [14]:

$$E_H(S) = \sqrt{\frac{Area(S)}{16\pi G^2}}[1+\frac{1}{2\pi}\oint_S(\rho\rho' - \frac{1}{8}\sigma_{ab}\overline{\sigma}^{ab} - \frac{1}{2}\omega_a\omega^a)dS], \quad (14)$$

where $\sigma^{ab}$ and $\overline{\sigma}^{ab}$ are shears and $\omega_a$ is the normal fundamental form. The energy is zero for any surface in flat spacetime, and reduces to the Hawking mass in the absence of shear and twist. For asymptotically flat spacetimes, the energy tends to the Bondi mass at null infinity and to the ADM mass at spatial infinity. It depends, however, implicitly on the gauge choice [2].

### D. Brown-York mass

Looking into the Hamiltonian formulation of general relativity, Brown and York [15] found interesting local quantities from which the definition of quasilocal mass was extracted. Consider a 3-dimensional spacelike slice $\Sigma$ bounded by a two-surface $B$ in a spacetime region that can be decomposed as a product of a spatial three-surface and a real line-interval representing the time. The time evolution of the two-surface boundary $B$ is the timelike three-surface boundary ${}^3B$. When $\Sigma$ is taken to intersect ${}^3B$ orthogonally, the Brown-York (BY) quasilocal energy is defined as:

$$E = \frac{1}{8\pi}\oint_B d^2x\sqrt{\sigma}(k - k_0), \quad (15)$$

where $\sigma$ is the determinant of the 2-metric on $B$, $k$ is the trace of the extrinsic curvature of $B$, and $k_0$ is a reference term that is used to normalize the energy with respect to a reference spacetime, not necessarily flat. This quasi local mass has the right asymptotic behavior but is not positive in general. The BY mass depends, however, on the choice of the gauge along the 3-dimensional spacelike slice $\Sigma$. To avoid this gauge dependence , Yau [18] (see also [16]and [17]) introduced a mass which is gauge independent and always positive. Based on Yau's definition, Liu-Yau defined the mass

$$E = -\frac{1}{8\pi}\oint_B d^2x\sqrt{\sigma}(\sqrt{k^2 - \ell^2} - k_0), \quad (16)$$

where $l$ and $k$ are traces of extrinsic curvatures $l_{ab} = \sigma_a^c\sigma_b^d\nabla_c u_d$ and $k_{ab} = \sigma_a^c\sigma_b^d\nabla_c n_d$ respectively, for $\sigma_{ab} = g_{ab} + u_a u_b - n_a n_b$ being the metric on the 2-sphere B. It was pointed out [20] that this Liu-Yau mass is strictly positive, even when the surface is in a flat spacetime.
It can be shown that for static spherically symmetric spacetimes the Brown-York quasilocal energy at the singularity is zero [22], in contrast to to the Newtonian gravity, in which the energy of the gravitational field diverges at the center for a point particle. Apparently, the nonlinearity of general relativity has removed this infinity. This QLE attains its maximum inside the horizon having an infinite derivative just on the horizon, before matching to its value outside the horizon: the black hole looks like an extended object.

## III. LTB METRIC

The LTB metric may be written in synchronous coordinates as

$$ds^2 = dt^2 - \frac{R'^2}{1+f(r)}dr^2 - R(t,r)^2 d\Omega^2. \quad (17)$$

It represents a pressure-less perfect fluid satisfying

$$\rho(r,t) = \frac{2M'(r)}{R^2 R'}, \qquad \dot{R}^2 = f + \frac{2M}{R}. \quad (18)$$

Here dot and prime denote partial derivatives with respect to the parameters $t$ and $r$, respectively. The angular distance $R$, depending on the value of $f$, is given by

$$R = -\frac{M}{f}(1 - \cos(\eta(r,t))),$$
$$\eta - \sin(\eta) = \frac{(-f)^{3/2}}{M}(t - t_n(r)), \quad (19)$$

for $f < 0$, and

$$R = (\frac{9}{2}M)^{\frac{1}{3}}(t - t_n)^{\frac{2}{3}}, \quad (20)$$

for $f = 0$, and

$$R = \frac{M}{f}(\cosh(\eta(r,t)) - 1),$$
$$\sinh(\eta) - \eta = \frac{f^{3/2}}{M}(t - t_n(r)), \quad (21)$$

for $f > 0$.
The metric is covariant under the rescaling $r \to \tilde{r}(r)$. Therefore, one can fix one of the three free parameters of the metric, i.e. $t_n(r)$, $f(r)$, and $M(r)$.
This metric has two generic singularities: the shell focusing singularity at $R(t,r) = 0$, and the shell crossing one at $R'(t,r) = 0$. However, if $\frac{M'}{R^2 R'}$ and $\frac{M}{R^3}$ are finite at $R = 0$ then there is no shell focusing singularity. Similarly, if $\frac{M'}{R'}$ is finite at $R' = 0$ then there is no shell crossing singularity. To get rid of the complexity of the shell focusing singularity, corresponding to a non-simultaneous big bang singularity, we may assume $t_n(r) = 0$, which will be the case for our toy models. This will enable us to concentrate on the behaviour of the collapse of an overdensity region in an expanding universe without interfering with the complexity of the inherent bang singularity of the metric [6].
It is easy to show that $\theta_{(\ell)}|_{R=2M} = 0$. Therefore, there may exist an apparent horizon, defined by $R = 2M$, being obviously a *marginally trapped tube*. It will turn out that this apparent horizon is not always spacelike and can have a complicated behaviour for different $r$ [6].

### A. Misner-Sharp mass

It is easily seen from (9) that $M(r)$ in the LTB metric is identical to the Misner-Sharp mass. The rate of change of this mass for any $R = const$ in the collapsing region is positive as can be seen by the following argumentation. Noting that $\dot{R} < 0$ in the collapsing region, and assuming no shell crossing, $R' > 0$, we obtain from $R'dr + \dot{R}dt = 0$ that $\frac{dr}{dt}|_{R=const} > 0$. Therefore, given $\frac{dM(r)}{dr} > 0$, we see that $\frac{dM(r)}{dt}|_{R=const} = \frac{dM(r)}{dr}\frac{dr}{dt}|_{R=const} > 0$.

### B. Hawking and Hayward masses

The LTB null tetrad needed to calculate the Hawking mass is given by

$$\ell^\mu = (1, \frac{\sqrt{1+f}}{R'}, 0, 0),\ n^\mu = (\frac{1}{2}, -\frac{\sqrt{1+f}}{R'}, 0, 0), \quad (22)$$

and

$$m^\mu = \frac{1}{R\sqrt{2}}(0, 0, 1, \frac{i}{sin\theta}),\ \bar{m}^\mu = \frac{1}{R\sqrt{2}}(0, 0, 1, \frac{-i}{sin\theta}). \quad (23)$$

We then obtain for the Hawking mass

$$M_{Haw} = M(r), \quad (24)$$

i.e. it is equivalent to the Misner-Sharp mass, as expected. This is due to the vanishing of twist and shear in metrics being spherically symmetric for round 2-sphere. We also conclude that the Hayward mass is identical to Hawking mass for the LTB metrics.

### C. Brown-York mass

The 2-boundary $B$ maybe specified by $r = constant$ and $t = constant$. We then obtain for the trace $k$ of the extrinsic curvature $k_{ab}$ for LTB's metric $k = -\frac{2\sqrt{1+f}}{R}$. The Brown-York energy is then given by

$$M_{BY} = -R\sqrt{1+f} - Subtraction\ term. \quad (25)$$

The subtraction term is chosen to be the corresponding FRW term for the $t = constant$ slice, i.e. $-R\sqrt{1+f}|_{FRW}$. Now the rate of change of the Brown-York mass is given by

$$\frac{dM_{BY}}{dt}|_{r=const} = -\dot{R}\sqrt{1+f} + (\dot{R}\sqrt{1+f})|_{FRW}. \quad (26)$$

The first term is responsible for the flow of dust falling into the center and the second term is due to the cosmological expansion. In the collapsing phase of the central region the first term is dominant and the Brown-York mass increases within the sphere of radius $r$.
The Brown-York mass may also be calculated for the 2-boundary $B$ with the constant physical radius $R = constant$ at $t = constant$, leading to

$$M_{BY}|_{R=const} = -R\sqrt{1 - \frac{2M}{R}} - Subtraction\ term. \quad (27)$$

The subtraction term is again the corresponding FRW term as the background.
Similarly, the Liu-Yau mass for the LTB black hole model is given by

$$M_{LY} = -R\sqrt{1 - \frac{2M}{R}} - Subtraction\ term, \quad (28)$$

which is valid for $k > \ell$, or for the untrapped region $R > 2m$ and Subtraction term$=-R$ . The rate of change of this mass is given by

$$\frac{dM_{LY}}{dt} = -\dot{R}\sqrt{1 - \frac{2M}{R}} - \frac{\dot{R}M}{R\sqrt{1 - \frac{2M}{R}}} - \frac{d(Subtraction\ term)}{dt}, \quad (29)$$

which is again an increasing function within the collapsing region.

## IV. MASS OF EVOLVING BLACK HOLES WITHIN FRW UNIVERSE

We are now interested in the mass of a cosmological overdensity region evolving into a black hole in a FRW background. We first choose two toy models and look for the mass of structures they represent. This should give us an overall view of the different mass definitions and their differences. Then we go to a more realistic model starting with a given density profile and look for the model parameters and the corresponding masses. Our cosmological black hole is going to be modeled by a LTB solution representing a collapsing overdensity region at the center and a flat FRW far from the overdensity region [6]. The overdensity region may take part in the expansion of the universe at early times but gradually the expansion is reversed and the collapsing phase starts. For a more realistic model we assume the familiar NFW profile [23] for the overdensity region within a LTB model and look for its consequences as regards different mass definitions.

### A. Example I:Toy model I with $\lim_{r\to\infty} f(r) \to 0^-$; structure within an asymptotically closed-flat LTB metric

The model is defined by the requirement $f(r) < 0$ and $f(r) \to 0$ when $r \to \infty$, and $M(0) = 0$. In both LTB toy models we assume $t_b = 0$. Let us use the ansatz $f(r) = -re^{-r}$ leading to

$$M(r) = \frac{1}{a} r^{3/2}(1 + r^{3/2}),$$

where $a$ is a constant having the dimension $[a] = [L]^{-2}$ [6]. The constant $a$ is fixed by $at_0 = 3\pi/2$, corresponding to the collapsing mass condensation around $r = 0$ starting in the expanding phase of the bound LTB model. Equations (19) and (20) then lead to

$$R = \frac{\sqrt{r}(1 + r^{3/2})}{ae^{-r}}(1 - \cos\eta(r, t)),$$
$$\eta - sin(\eta) = \frac{e^{-\frac{3}{2}r}}{(1 + r^{3/2})} at. \quad (30)$$

Fig.(1) shows schematically the behavior of the curvature function $f(r)$ and the corresponding Brown-York mass for a sphere of constant co-moving radius $r = constant$.

### B. Example II: Toy model II with $\lim_{r\to\infty} f(r) \to 0^+$; structure within an asymptotically open-flat LTB metric

What would happen if we choose the curvature function $f(r)$ such that it is negative for small $r$ but tends to zero for large $r$ while it is positive? We still have a model which tends to a flat FRW at large distances from the center, having a density less than the critical one corresponding to an open FRW model. The model is defined by the ansatz $f(r) = -r(e^{-r} - \frac{1}{r^n + c})$ with $n = 2$ and $c = 20000$, leading to [6]

$$M(r) = \frac{1}{a} r^{3/2}(1 + r^{3/2}),$$

where $a$ is a constant having the dimension $[a] = [L]^{-2}$ and fixed by the requirement $at_0 = 3\pi/2$. Equations (19) and (20) then lead to

$$R = \frac{\sqrt{r}(1 + r^{3/2})}{a(e^{-r} - \frac{1}{r^2+20000})}(1 - \cos\eta(r, t)),$$
$$\eta - sin\eta = \frac{(e^{-r} - \frac{1}{r^2+20000})^{1.5}}{(1 + r^{3/2})} at, \quad (31)$$

for $f < 0$ and

$$R = \frac{\sqrt{r}(1 + r^{3/2})}{a(\frac{1}{r^2+20000} - e^{-r})}(\cosh\eta(r, t) - 1),$$
$$\eta - sinh\eta = \frac{(\frac{1}{r^2+20000} - e^{-r})^{1.5}}{(1 + r^{3/2})} at, \quad (32)$$

for $f > 0$.

Fig.(1) shows schematically the two Brown-York masses inside spheres of constant comoving radius $r$ for the closed-flat and open-flat cosmological black hole toy models. Note the negative values of the BY mass in the case of open-flat model at distances far from the central overdensity region. In the case of closed-flat model, the Brown-York mass behaves similar to the corresponding Schwarzschild mass [22]. In both cases the Brown-York mass is zero at the central singularity in contrast to the Misner-Sharp and Hawking and Hayward mass, and remains finite within and on the horizon.

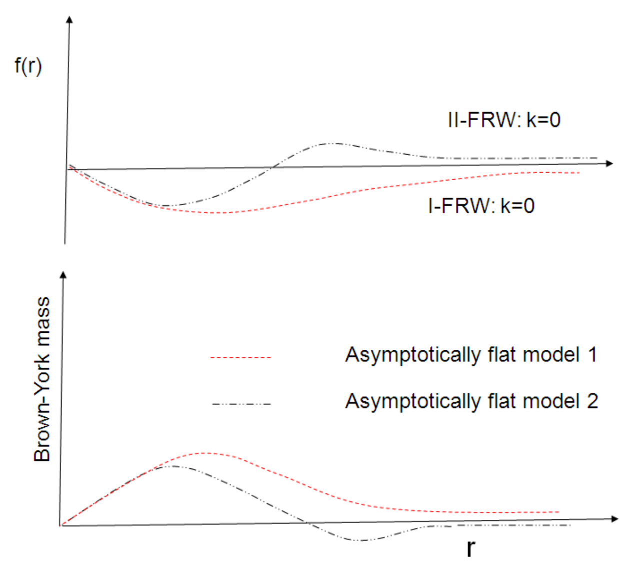


FIG. 1: The upper diagram shows curvature functions $f(r)$ for the closed-flat (I) and open-flat models (II). The lower diagram shows schematically two Brown-York masses for constant comoving radius $r$.

The more interesting case of the mass inside a sphere of constant physical radius $R$ is shown in Fig.(2) just for the closed-flat case. The corresponding mass behaviour for the open-flat case is similar to the closed-flat one, at least in the range of radius plotted in Fig.(2). Note the $\Omega-$value ($\Omega = \rho/\rho_{cr}$) as a function of $R$ plotted in the same figure. A comparison to the BY mass of the Schwarzschild metric [22] shows following similarities and differences. The Brown-York mass of our cosmological black hole toy model has two maxima: one near the apparent horizon and the other far from the horizon at cosmological distances, corresponding to the FRW cosmological horizon, and at the same time around a void occurring in the models studied. This void is, however, too shallow to be seen in the figure. We will see in the next section that in more realistic cases this second maximum splits in two different ones: one around the void where the density tends to the background value $\Omega = 1$, and the other around the cosmological horizon which is out of the range of our interest. The first maximum occurs just before the apparent horizon of the central black hole followed by an infinite slope similar to the BY mass of the Schwarzschild metric [22]. Let us call it the *Horizon mass maximum*, which can be seen to be related to the non-zero Misner-Sharp mass of the black hole. After reaching a minimum, the BY mass then increases again with increasing physical radius $R$ up to the second maximum, in contrast to the BY mass for the Schwarzschild case which decreases to the ADM mass. This second maximum, however, occurs after an infinite slope just after the cosmological particle horizon, in contrast to the behavior of the horizon mass maximum which has the infinite slope before the mass maximum and the apparent horizon. Because of the fact that this maximum occurs around the region where the density have reached the background FRW density, after passing a void and separating the central black hole from the almost homogeneous background, we call it the *structure mass maximum*. In contrast to the one corresponding to cosmological distances which we may call *cosmological mass maximum*. the separation of these two maxima will be obvious in the example III. Note that The Liu-Yau mass behaves similar to the Brown-York mass in the strong gravity region, but it approaches the Misner-Sharp mass far from the center.

Fig.(3) shows the BY mass at three different times, just before the onset of singularity and after the singularity has appeared, corresponding to different density profiles. It is obvious from the figure that the mass maxima increase and shift towards larger $R$ values as the time increases. The Horizon maximum is missing before the onset of the singularity. This maximum may be used as an indicator of singularity appearance in numerical relativity. The structure mass maximum is, however, present for any density profile irrespective of the occurrence of a central singularity (see Fig.10).

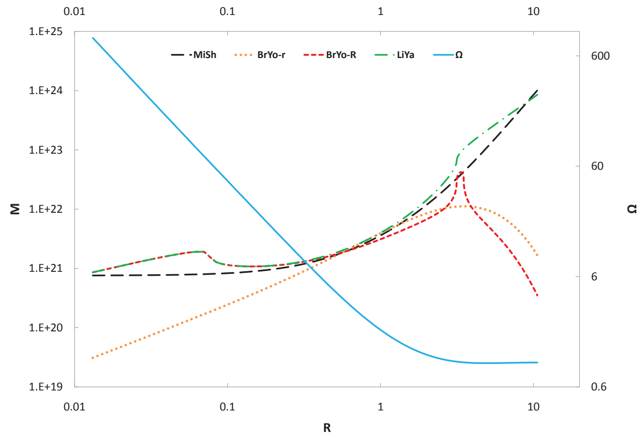


FIG. 2: Different masses of the cosmological black hole toy model I for constant physical radius $R$. Horizon radius is about 0.1 in terms of $R$ units.

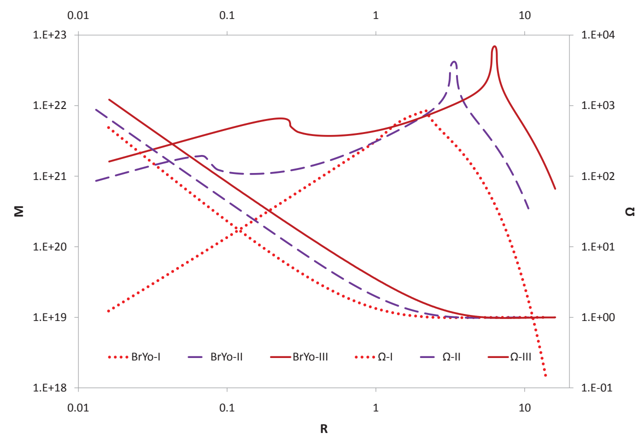


FIG. 3: Brown-York mass for three density profiles corresponding to three different times after the onset of the singularity and a time before the singularity has been formed. As the mass in-fall increases the black hole mass maximum increases too and shifts towards larger $R$ values. For dashed line, horizon radius is about 0.1 in terms of $R$ units.

### C. Example III: Structures within asymptotically flat LTB models having a NFW density profile

For a more realistic modeling of a cosmological structure, we need an algorithm to construct functions $M$, $f$, and $t_b$ in the LTB solution from physical quantities of the system such as the density profile. Krasińsky and Hellaby [24] propose an algorithm by which knowing the initial and final density profiles of an object one can find $f$ and $t_b$ as functions of $M$. For the sake of simplicity, it has been assumes here $M = r$.

We choose a Gaussian profile for the density at the last scattering surface as the initial time. The final profile, say at $z \sim 0.2$, is then chosen to be the universal density profile for the dark matter suggested by Navarro, Frenk and Wright [23]. To simulate a void compensating the overdensity mass region relative to the cosmological background [25][26], we convolute density profile of structure with a Gaussian underdensity. At far distances from the center of structure, density tends to the critical density, corresponding to a flat matter dominated cosmological background:

$$\rho_i(r) = \rho_{crit}(t_i)((\delta_{CMB}e^{-(\frac{r}{R_{i1}})^2} - b_1)e^{-(\frac{r}{R_{i2}})^2} + 1)$$
$$\rho_{NFW}(r) = \rho_{crit}\frac{\delta_c}{(\frac{r}{r_s})(1+\frac{r}{r_s})^2}$$
$$\rho_f(r) = \rho_{crit}(t_f)((\frac{\delta_c}{(\frac{r}{r_s})(1+\frac{r}{r_s})^2} - b_2)e^{-(\frac{r}{R_f})^2} + 1)$$

Using this algorithm, we have calculated the corresponding LTB functions, needed to define the metric and to calculate different masses. $f$ as a function of $M$ is depicted in Fig.(4). Note that, in contrast to toy models discussed above, now the bang time $t_b$ is non-vanishing as is shown Fig.(5). Theses functions have been calculated for $r$-values larger than a minimum corresponding to the central singularity $R = 0$. To calculate the mass we have assumed $r_s \approx 500kpc$ and $\delta_c \approx 4,000$ [27].

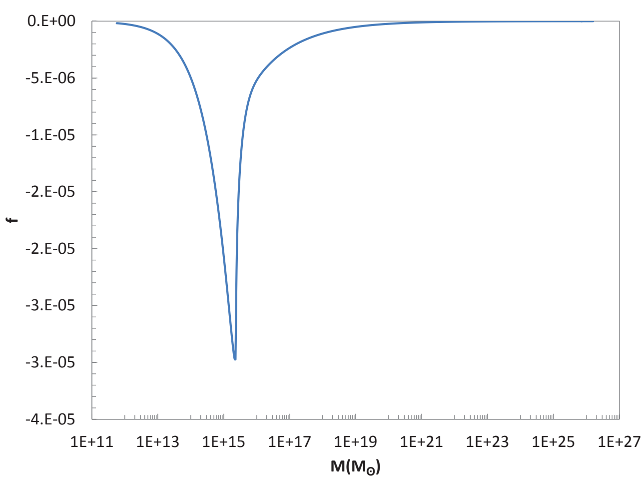


FIG. 4: $f(M)$ for a cluster of galaxies with NFW density profile in a flat background.

The result for different mass definitions is depicted in Fig.(6) as a function of the physical distance from the center. The mass up to the void turns out to be of the order of $10^{14}$ solar masses, which turns out interestingly to be of the same order for all mass definitions.

There are some interesting features in the Fig.(6). Note first that the horizon mass maximum, occurring also for the NFW profile, is not depicted in the Fig.(6) as it is very near to the origin of the figure. Numerical calculation shows that the apparent horizon is at a physical distance of the order of $10^{11}km = 0.01kpc$ corresponding to a Misner-Sharp mass of the order of $3 \times 10^{11}M_\odot$.

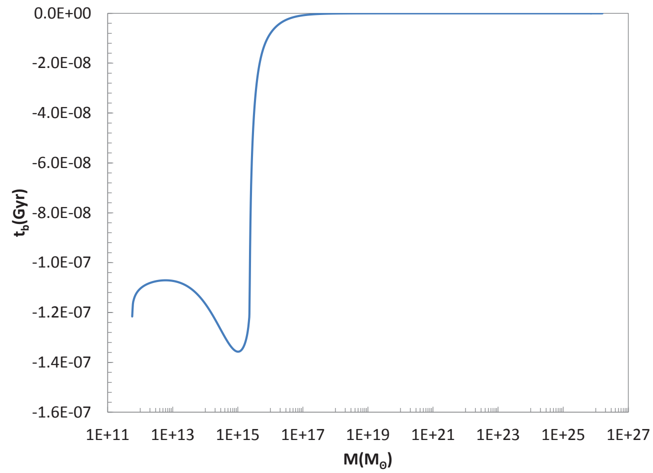


FIG. 5: $t_b(M)$ for a cluster of galaxies with NFW density profile in a flat background.

The structure maximum is shifted more to the left at distances of the order of few Mpc corresponding to the place of the void just before the density profile reaches the background values. Different mass definitions seems to coincide at this structure maximum which is about $10^{14}$ for the model we have constructed and it is about 300 times the horizon maximum mass and about 30 times the mass up to the distance of about 1Mpc, corresponding to $\Omega \approx 100$.
The *cosmological mass maximum* of Brown-York, which appears at cosmological distances, is now separated from the structure mass maximum. It is interesting to note that BY mass for constant co-moving and physical radius remains almost the same for distances after the void, while the Misner-Sharp and Liu-Yau masses increase, being almost equal. Fig.(7) shows the corresponding diagram for galactic masses. We recognize similar features as those for cluster masses, except the less exposed void at distances less than one Mpc.

The rate of matter flux through the apparent horizon (as a quasi-local black hole boundary in an asymptotically flat universe) is a useful quantity in astrophysical studies of black holes as well as theoretical study of black hole laws [28]. The matter flux for Misner-Sharp mass along the apparent horizon is given by $\frac{dM(r)}{dt}|_{AH} = \frac{dM(r)}{dr}\frac{dr}{dt}_{AH}$. As we see in Fig.(8), the rate of the matter flux increases with time at the initial phase of the black hole formation, up to a maximum value of the order of magnitude $10^6$ solar masses. It then decreases while the collapse is continuing and the black hole boundary (apparent horizon) freezes out in the expanding background (see Fig.9). In the case of BY mass, the rate of matter flux is similar to that of Misner-Sharp one (see Fig.(8)), being almost twice as much on the horizon as in the former case.

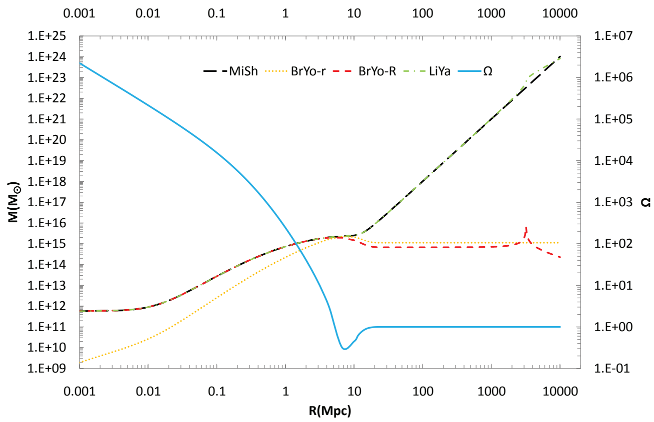


FIG. 6: Misner-Sharp, Brown-York, and Liu-Yau masses for a cluster of galaxies with NFW density profile in a flat background. The total density parameter $\Omega = \rho/\rho_{cr}$ as a function of the physical radius is also shown. Horizon radius is about $0.01pc$.

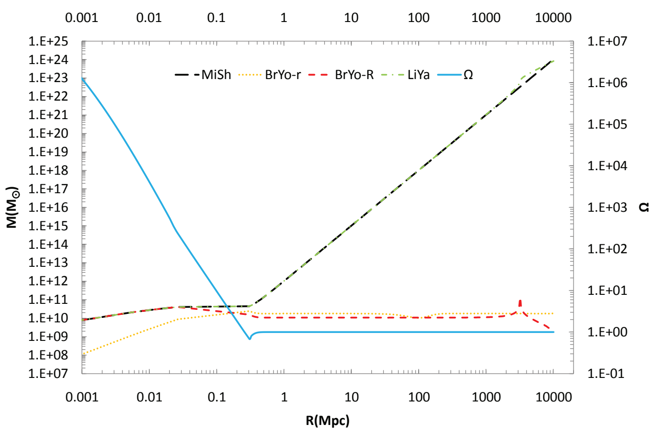


FIG. 7: Misner-Sharp, Brown-York and Liu-Yau masses for a galaxy with NFW density profile in a flat background. The density parameter $\Omega = \rho/\rho_{cr}$ is also shown. Horizon radius is about $10^{-4}pc$.

Fig.(10) compares the BY mass for two different NFW density profiles at a given time after the onset of the singularity. It shows the increase of the horizon and black hole or structure masses, and their shift to the right with the increasing density profile.

## V. DISCUSSION AND CONCLUSIONS

Being faced with the challenge to define mass of structures in a dynamical setting within general relativity, we have used models of mass condensation within a dynamical cosmological background to gain concrete insights of the similarities and differences between some

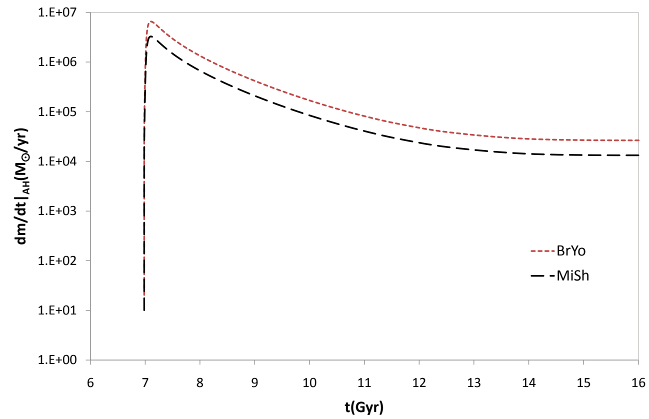


FIG. 8: The Misner-Sharp and Brown-York matter flux computed along apparent horizon. Note the rapid increase of the matter flux rate up to the 'freezing point' point of the Horizon and the following decrease.

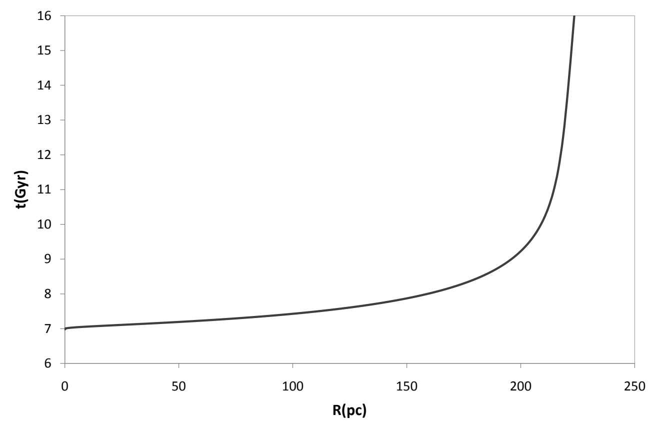


FIG. 9: The apparent horizon line in t-R plane.

of the more familiar mass definitions. The first two toy models representing collapsed overdense regions within asymptotically open-flat and closed-flat FRW models, show similar behavior at distances not very far from the collapsed region. The Brown-York mass, however, becomes negative for the open-flat model (Fig.1) at large distances relative to the place of the void where the density reaches the background one. For both models the BY mass tends to zero at infinity, as expected. It seems that independent of any density profile as the initial condition, if we wait enough, there is always a void before the density reaches the background value. At about the same distance, there is always a maximum of the mass, the 'structure mass maximum', which increases and move to the larger distances from the center as the density profile increases with the time through more infall of matter to the singularity. At the central singularity for $R = 0$ the Brown-York mass is zero in accordance with the BY mass of the Schwarzschild

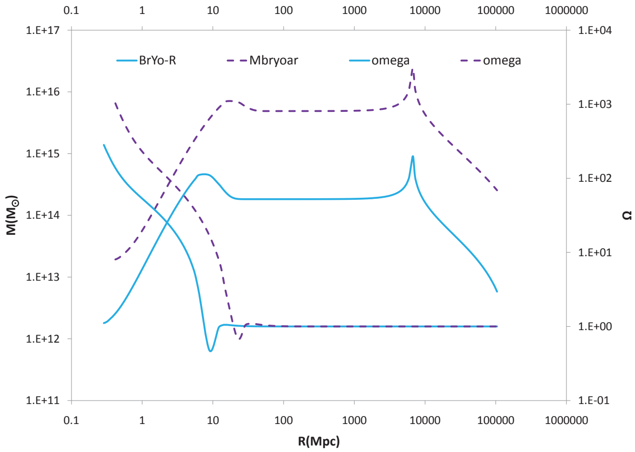


FIG. 10: Behavior of Brown-York mass for two different density profiles at the same time.

metric but in contrast to the Misner-Sharp, Hawking, and Hayward mass. At distances less than the void the BY mass is less than the Misner-Sharp mass. It is interesting to note that all mass definitions lead to almost the same value near the structure maximum.

In the case of the more realistic NFW density profile these features are even more distinguished. As can be seen from the Fig.(6), the structure maximum moves to the left at distances of the order of few Mpc for masses of the order of clusters. This is the same physical distance at which all three masses almost coincide and are equal to each other. At distances above $10Mpc$ the LY and MS mass are almost equal except for a maximum LY mass at distances corresponding to the cosmological BY maximum mass and differing from the Misner-Sharp ones. The BY mass at distances larger than place of the void may be defined as a function of the comoving radius or physical radius, being almost the same. It differs, however, substantially from the LY and Misner-Sharp mass, and remains almost constant up to large cosmological distances before reaching the last cosmological maximum.

The results obtained so far is indicative enough that the mass definitions may differ substantially. It is not said, however, that it may have any impact on our astrophysical mass determinations. In fact the mass definition in astrophysics is not as trivial as it may seems in the Newtonian dynamics. It is not even clear that there is any need at all for the concept of 'mass' in any astrophysical or cosmological setting. We are currently using our models to see if one can see any discrepancy between the general relativistic mass definitions and the one in Newtonian dynamics. Given the the Newtonian approximation for weak fields in general relativity, it is a legitimate question if this limit may also be used at cosmological distances [29]. We have already seen how different Misner-Sharp mass is relative to BY one! To tackle such questions we are currently applying our model to gravitational lensing phenomena [7] and the rotation curves of point masses within dynamical structures to see any deviations from the Newtonian approximation. Specifically, we model a lens as a structure in the cosmological background and solve the geodesic equations numerically in a general relativistic framework using our model structure. Note that the structure maximum mass occurs at points where the density is of the order of $10^{-29} g/cm^3$ and the gravity is weak enough to assume the Newtonian approximation. It is, however, not clear that we can ignore the nonlinear effects of general relativity at such large distances [29].


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